J. Fluid Mech. (1977), *vol.* 83, *part* 1, *pp.* 129–140
Printed in Great Britain

The diffusion of scalar and vector fields by homogeneous stationary turbulence

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(Received 4 October 1976 and in revised form 21 January 1977)

An exact Eulerian formulation of the problem of diffusion of passive scalar and vector fields by a turbulent velocity field is obtained. It is shown that, in the short autocorrelation time limit, the diffusion equation is exact for any turbulence. For non-zero autocorrelation times the form of the first few correction terms to the diffusion equation is found. As a result of these corrections the diffusion of scalar, divergence-free and curl-free vector fields will be different. The calculations use the Kubo–Van Kampen–Terwiel technique and are carried out for zero ordinary diffusivity and for homogeneous, stationary, isotropic, incompressible, helical turbulence.

1. Introduction

The subject of this paper is the problem of the diffusion of passive scalar and vector fields by a turbulent velocity field. In spite of much attention this problem is still far from being fully understood. The nonlinear problem in which the convected field is allowed to react back on the turbulent flow is a much harder problem and will not be considered here.

The early authors (Batchelor 1959; Saffman 1963) sought to simplify the problem by replacing the exact equations by approximate models based on physical ideas. Related ideas were used by Parker (1971) to argue that passive scalar and vector fields diffuse in the same way. Many authors tackled the problem directly by using the mean-field or Bourret approximation (Bourret 1962*a*, *b*; Brissaud & Frisch 1974; Van Kampen 1976) to obtain a closed equation for the mean field. Others (Kraichnan 1961, 1966; Roberts 1961) used the more refined direct-interaction approximation, designed to avoid certain time secularities of conventional perturbation theory (Weinstock 1969; Van Kampen 1976; but see Brissaud & Frisch 1974). Parker's conclusions were subsequently criticized by Moffatt (1974) and Kraichnan (1976*a*). Both Parker and Moffatt used the Lagrangian description of the diffusion process. Recently, however, Kraichnan (1976*a*) suggested that the Eulerian description might be more useful and used it to point out the importance of helicity fluctuations in the diffusion process.

The purpose of this paper is the clarification of the mathematical structure of the problem of the diffusion of scalar and vector fields by a prescribed turbulent velocity field. This structure has hitherto remained obscure. The problem is essentially that of solving stochastic differential equations. At present there are two main methods available for solving these equations. The first, due to Weinstock (1969) and Balescu & Misguich (1975), is a generalization of Bourret's original method, and yields an integro-differential equation for the mean field. The second, due to Kubo (1963), Van Kampen (1974*a*, *b*) and Terwiel (1974), is essentially an expansion in terms of the cumulants of the stochastic operator, and results in a differential equation for the mean field. Neither of these methods has hitherto been applied to the present problem.

In this paper we obtain an exact Eulerian formulation of the problem in terms of an integro-differential equation for the mean field, following the method of Balescu & Misguich (1975). This equation can be solved by a perturbation method. For homogeneous isotropic non-helical turbulence in which the velocity correlation time can be neglected (the short autocorrelation time approximation), the perturbation series can be evaluated. We find, in agreement with Kraichnan (1968), that the mean field obeys a diffusion equation for any turbulence. For helical turbulence, an exact dynamo equation is obtained.

For non-zero correlation times the perturbation series can be put into the cumulant-expansion form by the method of Terwiel (1974). This procedure is valid in the case of zero ordinary diffusivity. In this case the mean field obeys a more complicated partial differential equation. In this equation different terms dominate depending on the length scale of interest. We calculate the first two correction terms to the diffusion equation, and find that the diffusion equation is valid only on sufficiently large scales (small wavenumbers), and then with the turbulent diffusivity replaced by a new, renormalized turbulent diffusivity. For a magnetic field the dynamo equation is also valid only for small wavenumbers. This time the mean helicity also has to be renormalized. Owing to this renormalization the diffusion of scalar and vector fields will differ. It is found that the diffusion of a scalar field is reduced, while the renormalized diffusivity of a magnetic field could actually be negative as a result of helicity fluctuations, in agreement with Kraichnan (1976*a*). The diffusion of a curl-free vector field is not affected by helicity fluctuations.

2. The master equation for stochastic differential equations

In this section we shall develop the basic theory used in solving the equation

$$[\partial/\partial t + L(t)]f(t) = 0, \tag{1}$$

where $L(t)$ is a stochastic operator, independent of $f(t)$. Let

$$L(t) = \bar{L}(t) + L'(t), \tag{2}$$

where $\bar{L}(t) \equiv \langle L(t)\rangle$ and $L'(t)$ is the fluctuating part of $L(t)$. The angular brackets denote an ensemble average. $f(t)$ is similarly decomposed. Following Weinstock (1969) we define the projection operator A, which takes the average of everything to its right, and the propagator $U_A(t, t_0)$ by the equations

$$[\partial/\partial t + (1-A)L(t)]\, U_A(t,t_0) = 0, \quad U_A(t_0,t_0) = 1. \tag{3}$$

By operating on (1) from the left by A and $1-A$ respectively, and eliminating $f'(t)$ from the resulting equations using (3), we obtain the exact master equation

$$[\partial/\partial t + \bar{L}(t)]\bar{f}(t) = -\langle L'(t)\, U_A(t,t_0) f'(t_0)\rangle + \int_{t_0}^{t} dt_1 \langle L'(t)\, U_A(t,t_1)\, L'(t_1)\rangle \bar{f}(t_1). \tag{4}$$

We now express the propagator $U_A(t,t_0)$ in terms of the propagator $U_0(t,t_0)$ defined by

$$[\partial/\partial t+\bar{L}(t)]\,U_0(t,t_0)=0,\quad U_0(t_0,t_0)=1. \tag{5}$$

This propagator will be assumed to be known. From (3) we obtain

$$(\partial/\partial t+\bar{L})\,U_A=(\bar{L}A-(1-A)\,L')\,U_A. \tag{6}$$

Hence, using the definition of the propagator U_0, we obtain

$$U_A(t,t_0)=U_0(t,t_0)+\int_{t_0}^{t}dt_1\,U_0(t,t_1)\,[\bar{L}(t_1)\,A-(1-A)\,L'(t_1)]\,U_A(t_1,t_0). \tag{7}$$

This integral equation can be solved by a perturbation expansion and enables us to express the master equation (4) in terms of the propagator U_0:

$$U_A(t,t_0)=U_0(t,t_0)+\int_{t_0}^{t}dt_1\,U_0(t,t_1)\,[\bar{L}(t_1)\,A-(1-A)\,L'(t_1)]\,U_0(t_1,t_0)+\dots. \tag{8}$$

Using the identity $A^2=A$, the general term in this expression is found to be of the form

$$QA+(-1)^n\,U_0(1-A)\,L'U_0(1-A)\,L'\dots U_0(1-A)\,L'U_0, \tag{9}$$

where Q is an operator. On substituting into (4) the terms involving the Q's drop out. We obtain

$$[\partial/\partial t+\bar{L}(t)]\bar{f}(t)=\int_{t_0}^{t}dt'\Big\langle L'(t)\times\exp_0\Big\{-\int_{t'}^{t}dt_1\,U_0(t,t_1)\,(1-A)\,L'(t_1)\Big\}\,U_0(t_1,t')\,L'(t')\Big\rangle\bar{f}(t'), \tag{10}$$

where the subscript zero on the exponential denotes the usual time ordering. In writing (10) we have dropped the initial-value contribution because it may be assumed that $f'(t_0)$ is uncorrelated with the velocity field at all later times, or that it is zero. This assumption is convenient if we are interested in the long-term behaviour of the solution.

Equation (10) is exact. Equivalent equations have been given by Terwiel (1974) and Balescu & Misguich (1975).

3. The short autocorrelation time approximation

In general the expression (10) is very hard to evaluate, particularly as there is no theory that could be used to calculate all the moments of the turbulent velocity field. However, (10) can be evaluated easily in the short autocorrelation time approximation.

Consider the quantities b_n defined by

$$b_n(t_1,t_2,\dots,t_n)=(-1)^n\langle L'(t_1)\,U_0(t_1,t_2)\,(1-A)\,L'(t_2)\,U_0(t_2,t_3)\dots U_0(t_{n-1},t_n)\,(1-A)\,L'(t_n)\rangle, \tag{11}$$

where $t_1>t_2>t_3\dots>t_n$. It can be shown (Terwiel 1974) that the quantities b_n have the *cluster property*, namely that $b_n\simeq 0$ whenever the arguments $t_1,\dots,t_n$ can be divided into two or more groups in such a way that for t_i and t_j belonging to different groups one has $|t_i-t_j|\gtrsim\tau_c$, where τ_c is the autocorrelation time of $L'(t)$. We conclude that b_n is non-zero only when

$$t_1-t_n\lesssim(n-1)\,\tau_c. \tag{12}$$

In the short autocorrelation time approximation $\tau_c \to 0$. Let

$$b_n(t_1, \ldots, t_n) \propto \delta(t_1 - t_n). \tag{13}$$

Substituting the approximation (13) into (10), we obtain

$$(\partial/\partial t + \bar{L})\bar{f}(t) = \tau\langle L'(t)\,L'(t)\rangle \bar{f}(t), \tag{14}$$

where the integral correlation time τ is defined by

$$\langle L'(t)\,U_0(t-t_1)\,L'(t_1)\rangle = \tau\langle L'(t)\,L'(t)\rangle\,\delta(t-t_1). \tag{15}$$

Thus only the first term in (10) contributes in this limit. This is in agreement with the result of Kraichnan (1968). Equation (14) can be evaluated for the case of turbulent diffusion of scalar and vector fields.

A scalar field

The equation describing the turbulent diffusion of a passive scalar field is

$$[\partial/\partial t - \kappa\nabla^2 + \mathbf{u}(\mathbf{x},t).\nabla]\,\phi(\mathbf{x},t) = 0, \tag{16}$$

where $\mathbf{u}(\mathbf{x},t)$ is a homogeneous turbulent velocity field with zero mean. Writing this equation in the form (1), we define the operators

$$\bar{L} = -\kappa\nabla^2, \quad L' = \mathbf{u}(\mathbf{x},t).\nabla. \qquad (17), (18)$$

For isotropic incompressible turbulence (14) becomes

$$[\partial/\partial t - \kappa\nabla^2]\langle\phi(\mathbf{x},t)\rangle = \eta(t)\,\tau\nabla^2\langle\phi(\mathbf{x},t)\rangle, \tag{19}$$

where

$$\langle u_i(\mathbf{x},t)\,u_j(\mathbf{x},t)\rangle = \eta(t)\,\delta_{ij}, \quad \eta(t) > 0. \tag{20}$$

We thus have enhanced diffusion, the effective scalar diffusivity being increased to $\kappa + \eta\tau$. Equation (19) is independent of the helicity of the turbulence.

A divergence-free vector field

The equation describing the turbulent diffusion of a passive divergence-free vector field is the induction equation

$$\frac{\partial B_i}{\partial t} - \lambda\nabla^2 B_i + \left(u_k\frac{\partial}{\partial x_k}\delta_{ij} - \frac{\partial u_i}{\partial x_j}\right)B_j = 0. \tag{21}$$

The above theory can be formally applied to this equation by taking the propagators and stochastic operators to be second-rank tensor operators. Thus, writing (21) in the form (1), we define

$$\bar{L}_{ij} = -\lambda\delta_{ij}\nabla^2, \tag{22}$$

$$L'_{ij} = \delta_{ij}\,u_k\,\partial/\partial x_k - \partial u_i/\partial x_j. \tag{23}$$

Equation (14) becomes

$$[\partial/\partial t - \lambda\nabla^2]\langle\mathbf{B}(\mathbf{x},t)\rangle = \tau\langle\mathbf{L}'(t).\mathbf{L}'(t)\rangle.\langle\mathbf{B}(\mathbf{x},t)\rangle. \tag{24}$$

For isotropic helical turbulence we have, in addition to (20), the relation

$$\langle u_i(\mathbf{x},t)\,\partial u_j(\mathbf{x},t)/\partial x_k\rangle = \tfrac{1}{6}h(t)\,\epsilon_{ikj}, \tag{25}$$

where h is the mean helicity. Using incompressibility we obtain

$$\langle L'_{ij}\,L'_{jk}\rangle = \eta(t)\,\nabla^2\delta_{ik} + \tfrac{1}{3}h(t)\,\epsilon_{jik}\,\partial/\partial x_j, \tag{26}$$

so that (24) becomes

$$\partial\langle \mathbf{B}(\mathbf{x},t)\rangle/\partial t = (\lambda+\eta\tau)\,\nabla^2\langle \mathbf{B}(\mathbf{x},t)\rangle - \tfrac{1}{3}h\tau\nabla\times\langle \mathbf{B}(\mathbf{x},t)\rangle. \tag{27}$$

This is the usual dynamo equation (Moffatt 1970) thought to describe the amplification of magnetic fields in astrophysical objects. We have shown that it is exact for any turbulence in the short autocorrelation time approximation. For stationary turbulence in which η and h are independent of time, (27) has the exact Fourier-transformed solution (Vainshtein 1970; Moffatt 1970)

$$\langle \mathbf{B}(\mathbf{k},t)\rangle = \exp[-(\lambda+\eta\tau)\,k^2t]\,[\mathbf{B}(\mathbf{k},0)\cosh(\tfrac{1}{3}kh\tau t) - (i/k)\,\mathbf{k}\times\mathbf{B}(\mathbf{k},0)\sinh(\tfrac{1}{3}kh\tau t)], \tag{28}$$

showing that dynamo amplification can occur only on scales such that

$$2\pi/k > 6\pi(\lambda+\eta\tau)/|h|\tau. \tag{29}$$

This minimum scale has to be smaller than a typical dimension of the system.

Equation (27) was obtained by Kazantsev (1968) and Vainshtein (1970) for Gaussian turbulence by a method involving the summation of time-ordered reducible graphs. However, they did not implement the short autocorrelation time approximation consistently because they had b_n proportional to a product of n time δ-functions, instead of just one. In the short autocorrelation time limit only the first graph is present, and the result is valid for any turbulence. A similar error was made by Vainshtein (1972) and Vainshtein & Zel'dovich (1972).

In the absence of helicity, (19) and (27) are the same.

A curl-free vector field

A passive curl-free vector field is proportional to the vector representing a material surface element. Its turbulent diffusion is therefore described by the equation

$$\frac{\partial G_i}{\partial t} + u_j\frac{\partial G_i}{\partial x_j} = -G_j\frac{\partial u_j}{\partial x_i} + \lambda\nabla^2 G_i \tag{30}$$

(Batchelor 1967, p. 132). Writing this equation in the form (1), we have

$$\bar{L}_{ik} = -\lambda\delta_{ik}\nabla^2 \tag{31}$$

and

$$L'_{ik} = \delta_{ik}u_j\,\partial/\partial x_j + \partial u_k/\partial x_i. \tag{32}$$

Equation (24) can be evaluated for isotropic incompressible helical turbulence. Noting that $\nabla\times\mathbf{G} = 0$, the result is found to be the same as (19), again independent of the helicity.

4. Finite autocorrelation time

In this section we shall discuss the properties of the solution (10) in the general and realistic case of non-negligible autocorrelation time. In order to demonstrate the differences between the diffusion of scalar and vector fields, we shall for simplicity consider the case of zero ordinary diffusivity, so that the operator $\bar{L}$ vanishes and the propagators U_0 can be replaced by unity. Equation (10) becomes

$$\frac{\partial}{\partial t}\bar{f}(t) = \int_{t_0}^{t} dt'\langle L'(t)\exp_0\left\{-\int_{t'}^{t} dt_1(1-A)\,L'(t_1)\right\}L'(t')\rangle\bar{f}(t'). \tag{33}$$

This equation may be written as a cumulant expansion by the method of Terwiel (1974) and Van Kampen (1974*b*). We obtain (with $t_0 = 0$)

$$\frac{\partial}{\partial t}\bar{f}(t) = \left\{\sum_{m=2}^{\infty}(-1)^m K_m(t)\right\}\bar{f}(t), \tag{34}$$

where
$$K_m(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-2}} dt_{m-1} \langle\langle\langle L'(t)\, L'(t_1) \ldots L'(t_{m-1})\rangle\rangle\rangle. \tag{35}$$

Here the triple brackets denote an ordered cumulant defined by the following four rules:

(i) Write down a sequence of m dots.

(ii) Partition them into non-empty subsequences $\langle\ldots\rangle$ by inserting angular brackets in all possible ways.

(iii) For each partition consisting of p subsequences supply a factor $(-1)^{p-1}$.

(iv) In each partition write $L'(t)$ on the first dot and any permutation of

$$L'(t_1), \ldots, L'(t_{m-1})$$

on the remaining dots subject to the condition that each subsequence is correctly time ordered. Finally, add up all such partitions.

For commuting operators these cumulants reduce to the usual ones. Observe that (34) is a differential equation rather than an integro-differential equation. From (35) we see that $K_m = O(|L'|^m \tau_c^{m-1})$, so that (34) is a series expansion in powers of the autocorrelation time.

We now apply the results (34) and (35) to turbulent diffusion. We evaluate, for homogeneous, stationary, isotropic, incompressible turbulence, the first three terms in (34):

$$\begin{aligned}\frac{\partial}{\partial t}\bar{f}(t) = \Bigg\{&\int_0^t dt_1 \langle L'(t)\, L'(t_1)\rangle - \int_0^t dt_1 \int_0^{t_1} dt_2 \langle L'(t)\, L'(t_1)\, L'(t_2)\rangle \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [\langle L'(t)\, L'(t_1)\, L'(t_2)\, L'(t_3)\rangle - \langle L'(t)\, L'(t_1)\rangle \langle L'(t_2)\, L'(t_3)\rangle \\ &- \langle L'(t)\, L'(t_2)\rangle \langle L'(t_1)\, L'(t_3)\rangle - \langle L'(t)\, L'(t_3)\rangle \langle L'(t_1)\, L'(t_2)\rangle]\Bigg\}\bar{f}(t)\end{aligned} \tag{36}$$

(Van Kampen 1974*a*; Terwiel 1974). The general properties of the full equation (34) will be considered in a future paper.

A scalar field

For the turbulent diffusion of a scalar field the operator $L'(t)$ is given by (18). Equation (36) can be simplified using homogeneity and incompressibility. Furthermore, we observe that for isotropic turbulence the three-velocity correlations do not contribute because the only isotropic tensor with three free indices is antisymmetric, and these velocity correlations are contracted into symmetric multiple derivatives. We obtain (see appendix)

$$\frac{\partial}{\partial t}\langle\phi(\mathbf{x},t)\rangle = \{(\eta_2 + \eta_3 + \eta_4)\nabla^2 + \mu_4 \nabla^2\nabla^2 + \ldots\}\langle\phi(\mathbf{x},t)\rangle, \tag{37}$$

where $$\eta_2 = \frac{1}{3}\int_0^t dt_1 \langle u_i u_i' \rangle, \tag{38a}$$

$$\eta_3 = -\frac{1}{3}\int_0^t dt_1 \int_0^{t_1} dt_2 \langle u_i u_j' \partial_j u_i'' \rangle, \tag{38b}$$

$$\eta_4 = -\frac{1}{3}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle \partial_j u_i u_j' u_k'' \partial_k u_i''' \rangle, \tag{38c}$$

$$\mu_4 = \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{\tfrac{1}{15}[\langle u_i u_i' u_j'' u_j''' \rangle + \langle u_i u_j' u_i'' u_j''' \rangle + \langle u_i u_j' u_j'' u_i''' \rangle]$$
$$-\tfrac{1}{9}[\langle u_i u_i' \rangle \langle u_j'' u_j''' \rangle + \langle u_i u_i'' \rangle \langle u_j' u_j''' \rangle + \langle u_i u_i''' \rangle \langle u_j' u_j'' \rangle]\}. \tag{38d}$$

We have used the notation $\mathbf{u} = \mathbf{u}(t)$, $\mathbf{u}' = \mathbf{u}(t_1)$, etc., and $\partial_j \equiv \partial/\partial x_j$. The derivatives are understood to operate only on the velocity immediately following.

Equation (37) should be compared with (19) for $\kappa = 0$. We observe that the correction terms due to the non-vanishing of the autocorrelation time have two distinct effects. First, they alter the coefficient of the diffusion term. In general the n-velocity correlation will add a correction η_n to the turbulent diffusivity, as well as adding a correction to each of the higher-order terms ($\leqslant n$). Second, they change the diffusion equation into a more complicated partial differential equation by introducing higher spatial derivatives. We see therefore that the diffusion equation is a good approximation only for small wavenumbers. Even in this case, however, the usual turbulent diffusivity η_2 is replaced by a *renormalized* diffusivity η^* given by

$$\eta^* = \sum_{n=2}^{\infty} \eta_n. \tag{39}$$

On smaller scales the higher-order derivatives have to be taken into account.

A divergence-free vector field

For the turbulent diffusion of a magnetic field the operator L'_{ij} is given by (23). Equation (36) can again be simplified using homogeneity, incompressibility and isotropy. For helical turbulence we obtain (see appendix)

$$\partial \langle \mathbf{B}(\mathbf{x},t) \rangle / \partial t$$
$$= \{-(\alpha_2+\alpha_3+\alpha_4)\nabla \times + (\eta_2+\eta_3+\eta_4+\eta_4')\nabla^2 - \beta_4 \nabla \times \nabla^2 + \mu_4 \nabla^2\nabla^2 + \ldots\}\langle \mathbf{B}(\mathbf{x},t)\rangle. \tag{40}$$

The coefficients α_i, β_4 and η_4' are given in the appendix. Equation (40) is to be compared with (23). We observe again the general features noted in connexion with the scalar field diffusion. Thus for small wavenumbers (27) is a good approximation, provided the correct renormalized diffusivity is used, and provided also that the helicity is renormalized:

$$\alpha^* = \sum_{n=2}^{\infty} \alpha_n. \tag{41}$$

We observe, however, one additional important property of (40). In the absence of helicity (40) does not reduce to (37) owing to the presence of the term η_4'. This term is obtained from

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{\langle \partial_q u_p u'_q u''_s \partial_m u'''_i \rangle - \langle \partial_m u_i u'_p u''_r \partial_r u'''_s \rangle + \langle \partial_q u_p u'_q \partial_m u''_i u'''_s \rangle$$
$$- \langle u_p \partial_m u'_i u''_r \partial_r u'''_s \rangle + \langle \partial_m \partial_q u_i u'_q u''_p u'''_s \rangle - \langle u_p u'_s u''_r \partial_r \partial_m u'''_i \rangle$$
$$+ \langle u_p u'_s \partial_l u''_i \partial_m u'''_l \rangle + \langle u_p \partial_l u'_i u''_s \partial_m u'''_l \rangle + \langle u_p \partial_l u'_i \partial_m u''_l u'''_s \rangle$$
$$+ \langle \partial_l u_i u'_p u''_s \partial_m u'''_l \rangle + \langle \partial_l u_i u'_p \partial_m u''_l u'''_s \rangle + \langle \partial_l u_i \partial_m u'_l u''_p u'''_s \rangle\} \partial_p \partial_s \bar{B}_m \tag{42}$$

using isotropy and the condition $\partial_m \bar{B}_m = 0$. We obtain

$$\langle \partial_q u_p u'_q u''_s \partial_m u'''_i \rangle \partial_p \partial_s \bar{B}_m = -\tfrac{1}{15} \langle \partial_q u_p u'_q u''_s \partial_s u'''_p \rangle \nabla^2 \bar{B}_i \tag{43}$$

and

$$\langle \partial_l u_i \partial_m u'_l u''_p u'''_s \rangle \partial_p \partial_s \bar{B}_i = \tfrac{1}{15} [2 \langle \partial_l u_p \partial_p u'_l u''_s u'''_s \rangle - \langle \partial_l u_p \partial_s u'_l u''_p u'''_s \rangle] \nabla^2 \bar{B}_i. \tag{44}$$

The coefficient η'_4 can be calculated from (43) and (44) and their counterparts.

In order to investigate the significance of this coefficient we shall suppose that the velocity autocorrelation time is short so that the quantity (42) can be approximated by setting all the time arguments in the integrand equal. Then the first six terms cancel in pairs while the remaining six give a contribution

$$6 \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_m} u_p u_s \right\rangle. \tag{45}$$

This is essentially the two-helicity correlation. We conclude that as a result of the non-zero correlation (45) the turbulent diffusion of a passive scalar field will differ from that of a passive magnetic field, contrary to the statement of Parker (1971). Parker's proof, however, is in error because it is equivalent to keeping only the first term in the expansion (10). The quantity (45) can be expected to give a *negative* contribution to (40). To show this we approximate the quantity (45) using the assumption that the velocity field is normally distributed. From (44) we obtain for isotropic incompressible turbulence

$$\langle \partial_j u_i \partial_m u_j u_r u_s \rangle \partial_r \partial_s \bar{B}_m \simeq \tfrac{1}{15} [4 \langle \partial_l u_p u_s \rangle \langle \partial_p u_l u_s \rangle - \langle \partial_l u_p u_s \rangle \langle \partial_s u_l u_p \rangle] \nabla^2 \bar{B}_i$$
$$= -\tfrac{1}{18} h^2 \nabla^2 \bar{B}_i \tag{46}$$

using (25). Here h^2 denotes the mean-square helicity and will be positive even in turbulence with zero mean helicity. This result was obtained, somewhat heuristically, by Kraichnan (1976*a*). We note that in general the first six terms in the quantity (42) will also contribute to η'_4.

In the short autocorrelation time approximation η_4 is also negative definite, the quantity

$$\left\langle \frac{\partial u_i}{\partial x_j} u_j \frac{\partial u_k}{\partial x_l} u_l \right\rangle \tag{47}$$

being the mean-square *shear* correlation. The presence of these two terms raises the possibility that the renormalized turbulent diffusivity η^* could be negative, at least for sufficiently large times. This would cause amplification of the mean magnetic field (Kraichnan 1976*a*, *b*). Further differences between the diffusion of scalar and magnetic fields would be found on calculating further terms in the series (34).

A curl-free vector field

For the turbulent diffusion of a gradient the operator L'_{ij} is given by (32). Equation (36) can again be simplified using homogeneity, incompressibility and isotropy. Using the fact that **G** is a gradient we obtain (see appendix)

$$\partial\langle \mathbf{G}(\mathbf{x},t)\rangle/\partial t = \{(\eta_2+\eta_3+\eta_4)\nabla^2+\mu_4\nabla^2\nabla^2+\ldots\}\langle \mathbf{G}(\mathbf{x},t)\rangle, \tag{48}$$

which is independent of the mean helicity of the turbulence. Equation (48) can be obtained by taking the gradient of (37). This is because the vector field **G** is the gradient of a scalar field, so that its diffusion is described by the gradient of (16). Some general observations related to this problem have been made by Moffatt (1972).

5. Discussion

In the preceding section we calculated the first three terms in the differential equation describing the behaviour of the mean field in turbulent diffusion, under the assumption of zero normal diffusivity. They are the first three terms in a series expansion in powers of τ_c. The conditions for the convergence of the series (34) will be considered in a future paper.

The assumption of zero normal diffusivity is an important one, for it enables us to use the Van Kampen–Terwiel formalism, which is known to be free from secular terms (Van Kampen 1974*a*, 1976; Terwiel 1974). For example, one can easily demonstrate that it gives the exact answer to Bourret's (1965) pseudo-oscillator, as well as to other problems where the exact solutions are known. In general the Van Kampen–Terwiel technique can be extended to cases where there is a non-zero sure operator $\bar{L}$ as well as the stochastic operator L', by use of the interaction representation. However, for the problem of this paper this resource is not available, for the following reason. In the interaction representation (1) becomes

$$\partial\tilde{f}/\partial t = \tilde{L}'\tilde{f}, \tag{49}$$

where

$$\tilde{L}' = e^{Lt}L'e^{-Lt}, \quad \tilde{f} = e^{Lt}f. \tag{50}$$

With $\bar{L}$ defined by (17), (50) requires the definition of the operators $\exp(\kappa t\nabla^2)$ and $\exp(-\kappa t\nabla^2)$. The former is well defined because it is bounded in wavenumber space. In fact

$$\exp(\kappa t\nabla^2) = U_0(t) = (\kappa t)^{-\frac{3}{2}}\int d\mathbf{x}_0\exp\left(-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{\kappa t}\right). \tag{51}$$

The latter operator is, however, undefined because it is unbounded. This can be understood physically, because diffusion is an irreversible process, so that no inverse operator can exist. Hence, contrary to Van Kampen (1976), this method cannot be applied in the case where L is the diffusion operator.

When $\kappa > 0$, the method of §2 has to be used. This method is an extension of that due to Weinstock (1969) and Balescu & Misguich (1975). The Bourret (1962*a*, *b*) or mean-field approximation consists of keeping the first term in (10) and discarding the remainder. Although this method generally works (Brissaud & Frisch 1974), it can contain secular terms leading to the breakdown of the approximation for large times (Van Kampen 1976). However, one can argue, following Kraichnan (1976*a*), that if the velocity spectrum is peaked in wavenumber space at k_0, and f varies little on scales of k_0^{-1} and the eddy circulation time, then the smearing effect of the propagator U_0 will be negligible during the time τ_c. In that case the method of §4 can again be used, with the proviso that the renormalized diffusivity also includes κ.

The method of this paper enabled us to obtain a general solution to the problem of turbulent diffusion. Equations (10) and (34) show clearly that all the velocity correlations are required for a complete description of the diffusion process. Unfortunately at present they are of limited value because knowledge of these velocity correlations is lacking. However, our approach showed systematically that higher-order correlations are responsible for the differences between scalar and vector field diffusion. These differences cannot, therefore, be predicted within the Bourret approximation or the Kraichnan direct-interaction approximation. In this paper we have explicitly calculated the first three terms in a series solution in powers of the velocity autocorrelation time; these are sufficient to demonstrate a difference between the diffusion of these two kinds of field.

The author is grateful to Dr Robert Kraichnan, Professor David Layzer and Dr Robert Rosner for helpful discussions, and particularly to Professor H. K. Moffatt for his comments on the paper. This work was supported in part by the J. F. Kennedy Memorial Trust of Great Britain.

Appendix

We list here some results about velocity correlations in homogeneous incompressible turbulence used in deriving (37), (40) and (48). These results are an extension of those of Batchelor (1953) to correlations involving velocities at the same point but at several different times. We have

$$\langle u_i \partial_i u_j' \rangle = \partial_i \langle u_i u_j' \rangle = 0, \tag{A 1}$$

$$\langle \partial_j u_i \partial_k u_j' \rangle = \partial_j \langle u_i \partial_k u_j' \rangle = 0, \tag{A 2}$$

$$\begin{aligned}&\langle \partial_k u_i u_j' \partial_j \partial_l u_k'' \rangle - \langle \partial_j u_i \partial_k u_j' \partial_l u_k'' \rangle \\ &\quad = -\langle \partial_j \partial_k u_i u_j' \partial_l u_k'' \rangle - \langle \partial_j u_i \partial_k u_j' \partial_l u_k'' \rangle = -\partial_k \langle \partial_j u_i u_j' \partial_l u_k'' \rangle = 0,\end{aligned} \tag{A 3}$$

$$\langle \partial_i u_j u_k' \partial_k \partial_j u_l'' \rangle + \langle \partial_i u_j \partial_j u_k' \partial_k u_l'' \rangle = \partial_j \langle \partial_i u_j u_k' \partial_k u_l'' \rangle = 0, \tag{A 4}$$

$$\begin{aligned}&\langle \partial_j u_i \partial_k u_j' \partial_l u_k'' \partial_m u_l''' \rangle - \langle \partial_j u_i \partial_k u_j' u_l'' \partial_l \partial_m u_k''' \rangle - \langle \partial_j u_i u_k' \partial_k \partial_l u_j'' \partial_m u_l''' \rangle \\ &\quad - \langle \partial_j u_i u_k' \partial_l u_j'' \partial_m \partial_k u_l''' \rangle + \langle \partial_j u_i u_k' \partial_k u_l'' \partial_m \partial_l u_j''' \rangle + \langle \partial_j u_i u_k' u_l'' \partial_k \partial_l \partial_m u_j''' \rangle \\ &= \langle \partial_j u_i \partial_k u_j' \partial_l u_k'' \partial_m u_l''' \rangle - \langle \partial_j u_i \partial_l u_j' u_k'' \partial_k \partial_m u_l''' \rangle + \langle \partial_j \partial_k u_i u_k' \partial_l u_j'' \partial_m u_l''' \rangle \\ &\quad - \langle \partial_j \partial_k u_i u_k' u_l'' \partial_l \partial_m u_j''' \rangle \\ &= -\langle \partial_j u_i u_j' \partial_l u_k'' \partial_k \partial_m u_l''' \rangle - \langle \partial_j u_i \partial_l u_j' u_k'' \partial_k \partial_m u_l''' \rangle - \langle \partial_l \partial_j u_i u_j' u_k'' \partial_k \partial_m u_l''' \rangle = 0,\end{aligned} \tag{A 5}$$

$$\begin{aligned}&\langle \partial_i u_l u_q' \partial_q u_r'' \partial_r \partial_m u_m''' \rangle + \langle \partial_i u_l u_q' u_r'' \partial_q \partial_r \partial_l u_m''' \rangle + \langle \partial_i u_k u_q' \partial_q \partial_k u_l'' \partial_l u_m''' \rangle \\ &\quad + \langle \partial_i u_k u_q' \partial_k u_l'' \partial_q \partial_l u_m''' \rangle + \langle \partial_i u_j \partial_j u_k' u_l'' \partial_l \partial_k u_m''' \rangle + \langle \partial_i u_j \partial_j u_k' \partial_k u_l'' \partial_l u_m''' \rangle \\ &= -\langle \partial_i \partial_q u_k u_q' u_l'' \partial_l \partial_k u_m''' \rangle - \langle \partial_i \partial_q u_k u_q' \partial_k u_l'' \partial_l u_m''' \rangle - \langle \partial_i \partial_q u_k \partial_k u_q' u_l'' \partial_l u_m''' \rangle = 0,\end{aligned} \tag{A 6}$$

$$\langle u_p u_q' \partial_q u_r'' \partial_r u_s''' \rangle + \langle u_p u_q' u_r'' \partial_q \partial_r u_s''' \rangle = -\langle \partial_q u_p u_q' u_r'' \partial_r u_s''' \rangle. \tag{A 7}$$

For the pseudo-isotropic terms in (40) we need the result

$$-\langle u_p u'_q \partial_q \partial_l u''_i \rangle + \langle u_p \partial_l u'_k \partial_k u''_i \rangle - \langle \partial_l u_i u'_q \partial_q u''_p \rangle + \langle \partial_l u_k u'_p \partial_k u''_i \rangle + \langle \partial_l u_j \partial_j u'_i u''_p \rangle$$
$$= [\langle u_p \partial_l u'_k \partial_k u''_i \rangle + \langle \partial_l u_k u'_p \partial_k u''_i \rangle + \langle \partial_l u_k \partial_k u'_i u''_p \rangle] + [\langle \partial_q u_p u'_q \partial_l u''_i \rangle - \langle \partial_l u_i u'_q \partial_q u''_p \rangle]. \quad \text{(A 8)}$$

Finally we show that

$$[\langle u_p u'_r \partial_l u''_i \rangle + \langle u_p \partial_l u'_i u''_r \rangle + \langle \partial_l u_i u'_p u''_r \rangle] \partial_p \partial_r = 0. \quad \text{(A 9)}$$

To do this we follow Batchelor (1953, p. 53) and note that

$$\langle u_p(\mathbf{x}) u'_q(\mathbf{x}) u''_r(\mathbf{x}+\mathbf{r}) \rangle + \text{all permutations} = O(r^3) \quad \text{(A 10)}$$

in helical incompressible turbulence, because

$$\left\langle u_1 u'_1 \frac{\partial u''_1}{\partial x_1} \right\rangle + \left\langle u_1 \frac{\partial u'_1}{\partial x_1} u''_1 \right\rangle + \left\langle \frac{\partial u_1}{\partial x_1} u'_1 u''_1 \right\rangle = \frac{\partial}{\partial x_1} \langle u_1 u'_1 u''_1 \rangle = 0. \quad \text{(A 11)}$$

On using isotropy, η_4 follows from (A 7), η_3 from (A 9) and η_2 from (A 1). α_3 is obtained from (A 8), while

$$\alpha_2 \epsilon_{pki} = \int_0^t [\langle \partial_k u_i u'_p \rangle + \langle u_p \partial_k u'_i \rangle] \, dt_1; \quad \text{(A 12)}$$

$-\alpha_4$ is obtained from

$$\langle \partial_q u_p u'_q u''_r \partial_r \partial_m u'''_i \rangle + \langle \partial_m \partial_q u_i u'_q u''_r \partial_r u'''_p \rangle - \langle u_p \partial_k u'_i \partial_l u''_k \partial_m u'''_l \rangle$$
$$- \langle \partial_k u_i u'_p \partial_l u''_k \partial_m u'''_l \rangle - \langle \partial_j u_i \partial_l u'_j u''_p \partial_m u'''_l \rangle - \langle \partial_j u_i \partial_k u'_j \partial_m u''_k u'''_p \rangle$$
$$+ \langle \partial_j u_i \partial_m u'_j u''_r \partial_r u'''_p \rangle - \langle \partial_q u_p u'_q \partial_l u''_i \partial_m u'''_l \rangle + \langle u_p \partial_l u'_i u''_r \partial_r \partial_m u'''_l \rangle$$
$$+ \langle \partial_l u_i u'_p u''_r \partial_r \partial_m u'''_l \rangle + \langle \partial_q \partial_k u_i u'_q \partial_m u''_k u'''_p \rangle - \langle \partial_q \partial_l u'_q u''_p \partial_m u'''_l \rangle \quad \text{(A 13)}$$

and β_4 is obtained from

$$\langle u_p u'_q u''_r \partial_m u'''_i \rangle + \langle u_p u'_q \partial_m u''_i u'''_r \rangle + \langle u_p \partial_m u'_i u''_q u'''_r \rangle + \langle \partial_m u_i u'_p u'_q u'''_s \rangle. \quad \text{(A 14)}$$

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